25/10/23

MATH2050A Tutorial Planning

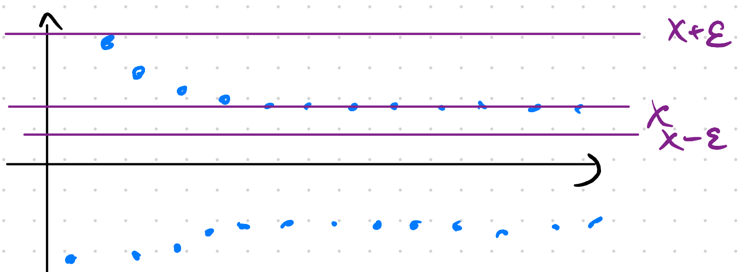
Recall Def: (x_n) bounded sequence.

1) $\limsup_{n\to\infty} x_n = \inf_{k\in\mathbb{N}} \sup_{n\geq k} x_n = \lim_{k\to\infty} \sup_{n\geq k} x_n$. "max of tail"

2) $\liminf_{n\to\infty} x_n = \sup_{k\in\mathbb{N}} \inf_{n\geq k} x_n = \lim_{k\to\infty} \inf_{n\geq k} x_n$. "min of tail".

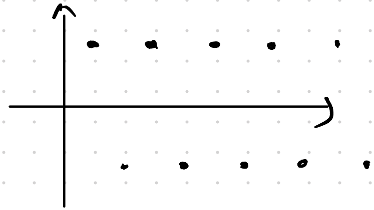
TFAE:

- $x = \limsup_{n\to\infty} x_n$
- for $\varepsilon>0$, there are $\overbrace{\text{at most finitely many } n \text{ s.t. } x+\varepsilon<x_n}^{\text{head part of the sequence}}$ but $\overbrace{\text{infinitely many } n \text{ s.t. } x-\varepsilon<x_n}^{\text{tail of the sequence.}}$.
- $x = \inf V$ where $V = \{v\in\mathbb{R} : v<x_n \text{ for at most finitely many } n\}$.
- $x = \sup S$ where $S = \{s\in\mathbb{R} : s = \lim_{k\to\infty} x_{n_k} \text{ for some } (n_k)\}$.



Thm: For a bounded sequence (x_n), x_n converges iff $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n$.

Ex: 1) $x_n = (-1)^n$.



$\limsup\limits_{n\to\infty} x_n = 1$, $\liminf\limits_{n\to\infty} x_n = -1$.

Q1: Alternate terms of the sequences $(1+\frac{1}{n})$, $(\frac{-1}{n})$ to obtain the sequence (x_n) given by

$$\left(2, -1, \frac{3}{2}, -\frac{1}{2}, 4/3, \dots\right).$$

Determine the values of $\limsup x_n$ and $\liminf x_n$. Also $\sup\{x_n\}$, $\inf\{x_n\}$.

Pf: Since $\frac{-1}{n} \le 1+\frac{1}{n}$, $\limsup\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} 1+\frac{1}{n} = 1$. $\sup x_n = 2$

$\liminf\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} -\frac{1}{n} = 0$. $\inf x_n = -1$.

Q2: (x_n), (y_n) bounded sequences. Then

$$\limsup (x_n + y_n) \le \limsup(x_n) + \limsup(y_n).$$

Give an example where the two sides are not equal.

Pf: $\limsup_{n\to\infty}(x_n+y_n)$: Let $v > \limsup(x_n)$, $u > \limsup(y_n)$. Then there are at most finitely many x_n s.t. $x_n > v$, at most finitely many y_n s.t. $y_n > u$. Then there are at most finitely many n s.t. $x_n + y_n > v + u$.

Then $\limsup_{n\to\infty}(x_n+y_n) \overset{(*)}{\le} v+u$. Then we get $\limsup_{n\to\infty}(x_n+y_n) \overset{(**)}{\le} \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$

(Take $v = \limsup_{n\to\infty} x_n + \varepsilon$, $u = \limsup_{n\to\infty} y_n + \varepsilon$, then $(*)$ means $\limsup_{n\to\infty}(x_n+y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n + 2\varepsilon$.
Since this is true for all $\varepsilon > 0$, we get $(**)$.)

Strict example: $x_n = (-1)^n$, $y_n = (-1)^{n+1}$.

Then $x_n + y_n = 0$, so $\limsup(x_n+y_n) = 0$, $\limsup_{n\to\infty} x_n = 1$, $\limsup_{n\to\infty} y_n = 1$. $0 < 2$.